Semantics proofs

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1 Preliminaries

Notations:

- φ, φ_1 and φ_2 denotes predicates.
- $\Diamond_{[a,b]}\varphi$ denotes the Eventually STL operator, evaluated on the time interval $[a,b]$.
- $\Box_{[a,b]}\varphi$ denotes the Always STL operator, evaluated on the time interval $[a,b]$.
- $\varphi_1\mathcal{U}_{[a,b]}\varphi_2$ denotes the Until STL operator, evaluated on the time interval $[a,b]$.
- τ denotes the current time.
- $\mathbb{T}$ denotes a set of discrete times such that $\mathbb{T} = \mathbb{N}$.
- $\mathbf{B}_\varphi$ indicates the offline version of an STL property φ which returns a Boolean value.
- $\mathbf{T}_\varphi^\tau$ indicates the positive version of an STL property φ, at time τ, which returns a Boolean value.
- $\mathbf{U}_\varphi^\tau$ indicates the indeterminate version of an STL property φ, at time τ, which returns a Boolean value.
- $\mathbf{F}_\varphi^\tau$ indicates the negative version of an STL property φ, at time τ, which returns a Boolean value.
- $\mathcal{X}$ denotes a finite set of signals.

For all the proofs, we consider the following assumption:

- Let $a,b \in \mathbb{T}$, we have $a < b$
- Every temporal operator is evaluated on the time interval $[t+a, t+b]$
- We consider only **non-nested** operators.

We admit the following result:

$$\exists t,\ (\mathcal{X},t) \vDash \varphi \iff \neg(\forall t,\ (\mathcal{X},t) \vDash \neg\varphi) \tag{1}$$

Remark 1 (Partitioning of time interval)**.** *Since, by assumption, $a < b$, we can partition the set of time $\mathbb{T}$ as $\mathbb{T} = [0;t+a[\,\cup\,[a,b[\,\cup\,[b,+\infty$. Using the current time τ, we can rely on the following terms: $(\tau < t+a) \vee (\tau \geq t+a \wedge \tau < t+b) \vee (\tau \geq t+b)$.*

2 Properties

2.1 Partitioning: completeness and disjointness

Property 1 (Complete and pairwise distinct)**.** *At any time instant t, exactly one of the three logic returns* `True` *for a given property.*

$$\mathbf{T}_\varphi^t \vee \mathbf{U}_\varphi^t \vee \mathbf{F}_\varphi^t \quad \textit{(completeness)} \tag{2}$$

$$\neg((\mathbf{T}_\varphi^t \wedge \mathbf{F}_\varphi^t) \vee (\mathbf{T}_\varphi^t \wedge \mathbf{U}_\varphi^t) \vee (\mathbf{U}_\varphi^t \wedge \mathbf{F}_\varphi^t)) \quad \textit{(disjointness)} \tag{3}$$

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2.2 Equivalence between online and offline logic

Property 2. *From a specific time instant t_f, the offline and online results are similar. Thus, the outputs of the offline $\mathbf{B}_\varphi$ and online $\mathbf{T}_\varphi^{t_f}$ versions are equivalent. In the same way, the negation of the offline operator is equivalent to the online negative version $\mathbf{F}_\varphi^{t_f}$. For a non-nested temporal operator evaluated on time interval $[a,b]$, this time instant corresponds at the latest to $t+b$:*

$$\tau \geq t+b \implies ((\mathbf{B}_\varphi \iff \mathbf{T}_\varphi^\tau) \wedge (\neg\mathbf{B}_\varphi \iff \mathbf{F}_\varphi^\tau)) \tag{4}$$

2.3 Operator determination

Remark 2 (Property determination)**.** *There exists an instant t_d from which we cannot satisfy $\mathbf{U}_\varphi^{t \geq t_d}$. For a non-nested temporal operator evaluated on time interval $[a,b]$, t_d corresponds at the latest to $t+b$.*

$$\exists t_d \leq t+b : \forall t' \geq t_d, \neg\mathbf{U}_\varphi^{t'} \tag{5}$$

Property 3 (Immutability: Positive and negative logics are final)**.** *If a property is satisfied in the positive (resp. negative) logic, it will remain so in the future.*

$$\exists t \in \mathbb{T} : \mathbf{T}_\varphi^t \implies \forall t' \geq t, \mathbf{T}_\varphi^{t'} \tag{6}$$

$$\exists t \in \mathbb{T} : \mathbf{F}_\varphi^t \implies \forall t' \geq t, \mathbf{F}_\varphi^{t'} \tag{7}$$

Remark 3. *Let $a,b,c \in \mathbb{T}$ and $c \geq b$.*

$$\exists t \in [a,b] : \varphi \implies \exists t \in [a,c] : \varphi \tag{8}$$

$$t' \geq t \implies t'+1 \geq t \tag{9}$$

3 Eventually

3.1 Eventually definitions

Eventually offline:

$$(\mathcal{X},t) \vDash \Diamond_{[a,b]}\varphi \Leftrightarrow \exists t' \in t+[a,b] : (\mathcal{X},t') \vDash \varphi \tag{10}$$

Eventually online: $P = \Diamond_{[a,b]}\varphi$

$$\mathbf{T}_P^\tau \quad \tau \geq t+a \wedge \exists t_1 \in [t+a, min(\tau,t+b)] : (\mathcal{X},t_1) \vDash \varphi \tag{11}$$

$$\mathbf{U}_P^\tau \quad (\tau < t+a) \vee (\tau < t+b \wedge \forall t_2 \in [t+a,\tau], (\mathcal{X},t_2) \vDash \neg\varphi) \tag{12}$$

$$\mathbf{F}_P^\tau \quad \tau \geq t+b \wedge \forall t_3 \in [t+a,t+b], (\mathcal{X},t_3) \vDash \neg\varphi \tag{13}$$

Let A, B and C such that:

$$A = \exists t_1 \in [t+a, min(\tau,t+b)] : (\mathcal{X},t_1) \vDash \varphi \tag{14}$$

$$B = \forall t_2 \in [t+a,\tau], (\mathcal{X},t_2) \vDash \neg\varphi \tag{15}$$

$$C = \forall t_3 \in [t+a,t+b], (\mathcal{X},t_3) \vDash \neg\varphi \tag{16}$$

We obtain:

$$\begin{array}{ll} \mathbf{T}_P^\tau & \tau \geq t+a \wedge A \\ \mathbf{U}_P^\tau & \tau < t+a \vee (\tau < t+b \wedge B) \\ \mathbf{F}_P^\tau & \tau \geq t+b \wedge C \end{array} \quad \begin{array}{r} (17) \\ (18) \\ (19) \end{array}$$

3.2 Completeness

Proof. Let us separate the consider time interval in three different sections as introduced above 1. We already showed that this separation includes all possible values of τ. Let us show that the property is satisfied for each of these three intervals.

1. **Starting hypothesis:** $\tau < t+a$

$$\begin{aligned} & \tau < t+a \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau) \\ \Longleftrightarrow\ & (\tau < t+a \wedge \mathbf{T}_P^\tau) \vee (\tau < t+a \wedge \mathbf{U}_P^\tau) \vee (\tau < t+a \wedge \mathbf{F}_P^\tau) \\ \Longleftrightarrow\ & (\tau < t+a \wedge \mathbf{T}_P^\tau) \vee ((\tau < t+a \wedge \tau < t+a) \vee (\tau < t+a \wedge \tau < t+b) \vee \\ & (\tau < t+a \wedge B)) \vee (\tau < t+a \wedge \mathbf{F}_P^\tau) \end{aligned}$$

According to the starting hypothesis, we can simplify as follows:

$$\begin{aligned} \Longleftrightarrow\ & (\tau < t+a \wedge \mathbf{T}_P^\tau) \vee True \vee (\tau < t+a \wedge \tau < t+b) \vee (\tau < t+a \wedge B) \vee \\ & (\tau < t+a \wedge \mathbf{F}_P^\tau) \\ \Longleftrightarrow\ & True \end{aligned} \quad (20)$$

2. **Starting hypothesis:** $\tau \geq t+a \wedge \tau < t+b$

$$\begin{aligned} & \tau \geq t+a \wedge \tau < t+b \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau) \\ \Longleftrightarrow\ & \tau \geq t+a \wedge \tau < t+b \wedge \left(\bigvee \begin{array}{l} (\tau \geq t+a \wedge A) \\ (\tau < t+a \vee (\tau < t+b \wedge B)) \\ (\tau \geq t+b \wedge C) \end{array} \right) \end{aligned}$$

using eqs. (17) to (19).

$$\Longleftrightarrow \tau \geq t+a \wedge \tau < t+b \wedge (A \vee B)$$

simplifying and removing false cases in disjunction.

$$\Longleftrightarrow \tau \geq t+a \wedge \tau < t+b \wedge \left(\bigvee \begin{array}{l} (\exists t_1 \in [t+a, min(\tau, t+b)], (\mathcal{X}, t_1) \vDash \varphi) \\ (\forall t_2 \in [t+a, \tau], (\mathcal{X}, t_2) \vDash \neg\varphi) \end{array} \right)$$

using eqs. (14) and (15)

$$\Longleftrightarrow \tau \geq t+a \wedge \tau < t+b \wedge \left(\bigvee \begin{array}{ll} \neg & (\forall t_1 \in [t+a, \tau], (\mathcal{X}, t_1) \vDash \neg\varphi) \\ & (\forall t_2 \in [t+a, \tau], (\mathcal{X}, t_2) \vDash \neg\varphi) \end{array} \right)$$

simplifying and replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\Longleftrightarrow True \quad (21)$$

3. **Starting hypothesis:** $\tau \geq t+b$

$$
\begin{aligned}
& \tau \geq t+b \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau) \\
\Longleftrightarrow & (\tau \geq t+b \wedge \mathbf{T}_P^\tau) \vee (\tau \geq t+b \wedge \mathbf{U}_P^\tau) \vee (\tau \geq t+b \wedge \mathbf{F}_P^\tau) \\
\Longleftrightarrow & (\tau \geq t+b \wedge \tau \geq t+a \wedge A) \vee (\tau \geq t+b \wedge \tau < t+a) \vee \\
& (\tau \geq t+b \wedge \tau < t+b \wedge B) \vee (\tau \geq t+b \wedge \tau \geq t+b \wedge C)
\end{aligned}
$$

According to (1), we simplify as follows:

$$
\begin{aligned}
\Longleftrightarrow & (\tau \geq t+b \wedge \tau \geq t+b \wedge A) \vee (\tau \geq t+b \wedge \tau < t+b) \vee \\
& (\tau \geq t+b \wedge \tau < t+b \wedge B) \vee (\tau \geq t+b \wedge \tau \geq t+b \wedge C) \\
\Longleftrightarrow & (\tau \geq t+b \wedge \tau \geq t+b \wedge A) \vee (\tau \geq t+b \wedge \neg(\tau \geq t+b)) \vee \\
& (\tau \geq t+b \wedge \neg(\tau \geq t+b) \wedge B) \vee (\tau \geq t+b \wedge \tau \geq t+b \wedge C) \\
\Longleftrightarrow & (\tau \geq t+b \wedge \tau \geq t+b \wedge A) \vee False \vee False \\
& \vee (\tau \geq t+b \wedge \tau \geq t+b \wedge C)
\end{aligned}
$$

According to the starting hypothesis, we simplify as follows:

$$
\begin{aligned}
\Longleftrightarrow & A \vee C \\
\Longleftrightarrow & \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi \vee \\
& \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi
\end{aligned}
$$

According to the starting hypothesis, we simplify $[t+a, min(\tau, t+b)]$ in $[t+a, t+b]$:

$$
\Longleftrightarrow \exists t_1 \in [t+a, t+b] : (\mathcal{X}, t_1) \vDash \varphi \vee \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi
$$

Then, we use (1):

$$
\begin{aligned}
\Longleftrightarrow & \neg(\forall t_1 \in [t+a, t+b], (\mathcal{X}, t_1) \vDash \neg\varphi) \vee \\
& \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi \\
\Longleftrightarrow & True
\end{aligned} \tag{22}
$$

We demonstrated that:

- The division into three time intervals that we have used covers all possible cases for τ (1);
- In each of these temporal intervals, there is always a version of the *Eventually* operator which is `True` whatever τ eqs. (20) to (22).

We have therefore shown that the completeness property 2 is satisfied by the *Eventually* semantics for any value of τ. □

3.3 Disjointness

Proof. Let us come back to the exclusivity property 3:

$$\begin{aligned} &\neg[(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \vee (\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \vee (\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau)] \\ \iff &\neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \wedge \neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \wedge \neg(\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau) \end{aligned} \tag{23}$$

Let us consider separately the three different cases and show that they are satisfied by the *Eventually* semantics.

- $\neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau)$

  $$\neg(\tau \geq t+a \wedge A \wedge \tau \geq t+b \wedge C)$$

  According to (1), we simplify the time conditions as follows:

  $$\begin{aligned} \iff &\neg(\tau \geq t+b \wedge A \wedge C) \\ \iff &\neg(\tau \geq t+b \wedge \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi \wedge \\ &\forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi) \\ \iff &\neg(\tau \geq t+b \wedge \exists t_1 \in [t+a, t+b] : (\mathcal{X}, t_1) \vDash \varphi \wedge \\ &\forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi) \end{aligned}$$

  Then, we use (1):

  $$\begin{aligned} \iff &\neg(\tau \geq t+b \wedge \neg(\forall t_1 \in [t+a, t+b], (\mathcal{X}, t_1) \vDash \neg\varphi) \wedge \\ &\forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi) \\ \iff &\neg(\tau \geq t+b \wedge False) \\ \iff &\neg False \\ \iff &True \end{aligned} \tag{24}$$

- $\neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau)$

  $$\begin{aligned} &\neg(\tau \geq t+a \wedge A \wedge (\tau < t+a \vee (\tau < t+b \wedge B))) \\ \iff &\neg((\tau \geq t+a \wedge A \wedge \tau < t+a) \vee (\tau \geq t+a \wedge A \wedge \tau < t+b \wedge B)) \\ \iff &\neg((\tau \geq t+a \wedge \neg(\tau \geq t+a) \wedge A) \vee (\tau \geq t+a \wedge \tau < t+b \wedge A \wedge B)) \\ \iff &\neg(False \vee (\tau \geq t+a \wedge \tau < t+b \wedge A \wedge B)) \\ \iff &\neg(\tau \geq t+a \wedge \tau < t+b \wedge A \wedge B) \\ \iff &\neg(\tau \geq t+a \wedge \tau < t+b \wedge \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi \wedge \\ &\forall t_2 \in [t+a, \tau], (\mathcal{X}, t_2) \vDash \neg\varphi) \\ \iff &\neg(\tau \geq t+a \wedge \tau < t+b \wedge \exists t_1 \in [t+a, \tau] : (\mathcal{X}, t_1) \vDash \varphi \wedge \\ &\forall t_2 \in [t+a, \tau], (\mathcal{X}, t_2) \vDash \neg\varphi) \end{aligned}$$

According to (1), we replace $\exists$ by $\forall$ as follows:

$$\begin{aligned}
&\iff \neg(\tau \geq t+a \wedge \tau < t+b \wedge \neg(\forall t_1 \in [t+a,\tau], (\mathcal{X},t_1) \vDash \neg\varphi) \wedge \\
&\quad \forall t_2 \in [t+a,\tau], (\mathcal{X},t_2) \vDash \neg\varphi) \\
&\iff \neg(\tau \geq t+a \wedge \tau < t+b \wedge False) \\
&\iff \neg False \\
&\iff True
\end{aligned} \tag{25}$$

- $\neg(\mathbf{U}^\tau_P \wedge \mathbf{F}^\tau_P)$

$$\begin{aligned}
&\neg((\tau < t+a \vee (\tau < t+b \wedge B)) \wedge \tau \geq t+b \wedge C) \\
\iff &\neg((\tau < t+a \wedge \tau \geq t+b \wedge C) \vee (\tau < t+b \wedge B \wedge \tau \geq t+b \wedge C))
\end{aligned}$$

According to (1), we simplify the time conditions as follows:

$$\begin{aligned}
&\iff \neg((\tau < t+b \wedge \tau \geq t+b \wedge C) \vee (\tau < t+b \wedge B \wedge \tau \geq t+b \wedge C)) \\
&\iff \neg((\tau < t+b \wedge \neg(\tau < t+b) \wedge C) \vee (\tau < t+b \wedge B \wedge \neg(\tau < t+b) \wedge C)) \\
&\iff \neg((False \wedge C) \vee (False \wedge B \wedge C)) \\
&\iff \neg(False \vee False) \\
&\iff \neg False \\
&\iff True
\end{aligned} \tag{26}$$

Now, we can comeback to the beginning equation 4.3:

$$\neg(\mathbf{T}^\tau_P \wedge \mathbf{F}^\tau_P) \wedge \neg(\mathbf{T}^\tau_P \wedge \mathbf{U}^\tau_P) \wedge \neg(\mathbf{U}^\tau_P \wedge \mathbf{F}^\tau_P) \tag{27}$$

Based on the results of eqs. (24) to (26), we obtain:

$$\begin{aligned}
&\iff True \wedge True \wedge True \\
&\iff True
\end{aligned} \tag{28}$$

We showed that the disjointness condition is satisfied by the *Eventually* semantics.

□

3.4 Equivalence between offline and online logic

Proof.

$$\begin{aligned}
&\tau \geq t+b \implies ((\mathbf{B}_P \Leftrightarrow \mathbf{T}^\tau_P) \wedge \neg(\mathbf{B}_P \Leftrightarrow \mathbf{F}^\tau_P)) \\
\iff &(\tau \geq t+b \implies (\mathbf{B}_P \Leftrightarrow \mathbf{T}^\tau_P)) \wedge (\tau \geq t+b \implies (\neg\mathbf{B}_P \Leftrightarrow \mathbf{F}^\tau_P))
\end{aligned}$$

Let us consider that $\tau \geq t+b$. Let us show that it implies that $(\mathbf{B}_P \Leftrightarrow \mathbf{T}^\tau_P)$ and $(\neg\mathbf{B}_P \Leftrightarrow \mathbf{F}^\tau_P)$ are *True*.

$$\begin{aligned}
\iff &\exists t' \in t+[a,b] : (\mathcal{X},t') \vDash \varphi \Leftrightarrow \\
&\tau \geq t+a \wedge \exists t_1 \in [t+a, min(\tau,t+b)] : (\mathcal{X},t_1) \vDash \varphi \wedge \\
&(\neg(\exists t' \in t+[a,b] : (\mathcal{X},t') \vDash \varphi \Leftrightarrow \\
&\tau \geq t+b \wedge \forall t_3 \in [t+a,t+b], (\mathcal{X},t_3) \vDash \neg\varphi)
\end{aligned}$$

From 1 and the temporal hypothesis $\tau \geq t+b$, we simplify as follows the time intervals and time conditions:

$$\begin{aligned}
&\Longleftrightarrow \exists t' \in [t+a, t+b] : (\mathcal{X}, t') \vDash \varphi \Leftrightarrow \\
&\quad \exists t_1 \in [t+a, t+b] : (\mathcal{X}, t_1) \vDash \varphi \wedge \\
&\quad (\neg(\exists t' \in [t+a, t+b] : (\mathcal{X}, t') \vDash \varphi) \Leftrightarrow \\
&\quad \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi) \\
&\Longleftrightarrow True \wedge (\neg(\exists t' \in [t+a, t+b] : (\mathcal{X}, t') \vDash \varphi) \Leftrightarrow \\
&\quad \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi)
\end{aligned}$$

According to (1), we replace $\exists$ by $\forall$ as follows:

$$\begin{aligned}
&\Longleftrightarrow \neg(\neg \forall t' \in [t+a, t+b] : (\mathcal{X}, t') \vDash \neg\varphi) \Leftrightarrow \\
&\quad \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi \\
&\Longleftrightarrow \forall t' \in [t+a, t+b] : (\mathcal{X}, t') \vDash \neg\varphi \Leftrightarrow \\
&\quad \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi \\
&\Longleftrightarrow True
\end{aligned} \tag{29}$$

□

3.5 Immutability: Positive and negative logics are final

Proof.

- $\exists t \in \mathbb{T} : \mathbf{T}_{\varphi}^{t} \implies \forall t' \geq t, \mathbf{T}_{\varphi}^{t'}$

  Directly proven according to Remark 3.

- $\exists t \in \mathbb{T} : \mathbf{F}_{\varphi}^{t} \implies \forall t' \geq t, \mathbf{F}_{\varphi}^{t'}$

  Let us come back to the negative logic equation:

$$\tau \geq t+b \wedge \forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi \tag{30}$$

  $\tau \geq t+b$ part is directly proven according to Remark 3.

  $\forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \neg\varphi$ is not affected by the current time so if the property was already True at a time t', it is still the case at time $t'+1$.

We showed that whatever the time t', if $\mathbf{T}_{P}^{t'}$ or $\mathbf{F}_{P}^{t'}$ is satisfied, it is still the case at time $t'+1$ □

4 Always

4.1 Always definitions

Always offline:

$$(\mathcal{X}, t) \vDash \square_{[a,b]} \varphi \Leftrightarrow \forall t' \in t + [a, b] : (\mathcal{X}, t') \vDash \varphi \tag{31}$$

Always online: $\Box_{[a,b]}\varphi$

$$\mathbf{T}_P^\tau \quad \tau \geq t+b \wedge \forall t_1 \in [t+a,t+b], (\mathcal{X},t_1) \vDash \varphi \tag{32}$$
$$\mathbf{U}_P^\tau \quad (\tau < t+a) \vee (\tau < t+b \wedge \forall t_2 \in [t+a,\tau], (\mathcal{X},t_2) \vDash \varphi) \tag{33}$$
$$\mathbf{F}_P^\tau \quad \tau \geq t+a \wedge \exists t_3 \in [t+a, min(\tau,t+b)] : (\mathcal{X},t_3) \vDash \neg\varphi \tag{34}$$
$$\tag{35}$$

Let A, B and C such that:

$$A = \forall t_1 \in [t+a,t+b], (\mathcal{X},t_1) \vDash \varphi \tag{36}$$
$$B = \forall t_2 \in [t+a,\tau], (\mathcal{X},t_2) \vDash \varphi \tag{37}$$
$$C = \exists t_3 \in [t+a, min(\tau,t+b)] : (\mathcal{X},t_3) \vDash \neg\varphi \tag{38}$$

We obtain:

$$\mathbf{T}_P^\tau \quad \tau \geq t+b \wedge A \tag{39}$$
$$\mathbf{U}_P^\tau \quad \tau < t+a \vee (\tau < t+b \wedge B) \tag{40}$$
$$\mathbf{F}_P^\tau \quad \tau \geq t+a \wedge C \tag{41}$$

4.2 Completeness

Proof. Let us separate the consider time interval in three different sections as introduced in the property 1. Let us show that the property is satisfied for each of these three intervals.

1. **Starting hypothesis:** $\tau < t+a$

$$\tau < t+a \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau)$$
$$\iff \tau < t+a \wedge \left(\bigvee \begin{array}{l} (\tau \geq t+b \wedge A) \\ (\tau < t+a) \\ (\tau < t+b \wedge B) \\ (\tau \geq t+a \wedge C) \end{array} \right)$$

using eqs. (39) to (41).

$$\iff True$$

by disjunction.

$$\tag{42}$$

2. **Starting hypothesis:** $\tau \geq t+a \wedge \tau < t+b$

$$\tau \geq t+a \wedge \tau < t+b \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau)$$
$$\iff \tau \geq t+a \wedge \tau < t+b \wedge \left(\bigvee \begin{array}{l} (\tau \geq t+b \wedge A) \\ (\tau < t+a) \\ (\tau < t+b \wedge B) \\ (\tau \geq t+a \wedge C) \end{array} \right)$$

using eqs. (39) to (41).

$$\iff \tau \geq t+a \wedge \tau < t+b \wedge (B \vee C)$$

simplifying and removing false cases in disjunction

$$\iff \tau \geq t+a \wedge \tau < t+b \wedge \left(\bigvee \begin{array}{c} (\forall t_2 \in [t+a,\tau],(\mathcal{X},t_2) \vDash \varphi) \\ (\exists t_3 \in [t+a,\tau] : (\mathcal{X},t_3) \vDash \neg\varphi) \end{array} \right)$$

simplifying and using eqs. (37) and (38).

$$\iff \tau \geq t+a \wedge \tau < t+b \wedge \left(\bigvee \begin{array}{c} (\forall t_2 \in [t+a,\tau],(\mathcal{X},t_2) \vDash \varphi) \\ \neg(\forall t_3 \in [t+a,\tau],(\mathcal{X},t_3) \vDash \varphi) \end{array} \right)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\iff True \tag{43}$$

3. **Starting hypothesis:** $\tau \geq t+b$

$$\tau \geq t+b \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau)$$

$$\iff \tau \geq t+b \wedge \left(\bigvee \begin{array}{l} (\tau \geq t+b \wedge A) \\ (\tau < t+a) \\ (\tau < t+b \wedge B) \\ (\tau \geq t+a \wedge C) \end{array} \right)$$

using eqs. (39) to (41).

$$\iff \tau \geq t+b \wedge (A \vee C)$$

$$\iff \tau \geq t+b \wedge \left(\bigvee \begin{array}{c} (\forall t_1 \in [t+a,t+b],(\mathcal{X},t_1) \vDash \varphi) \\ (\exists t_3 \in [t+a,t+b] : (\mathcal{X},t_3) \vDash \neg\varphi) \end{array} \right)$$

simplifying and using eqs. (36) and (38).

$$\iff \tau \geq t+b \wedge \left(\bigvee \begin{array}{c} (\forall t_1 \in [t+a,t+b],(\mathcal{X},t_1) \vDash \varphi) \\ \neg(\forall t_3 \in [t+a,t+b],(\mathcal{X},t_3) \vDash \varphi) \end{array} \right)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\iff True \tag{44}$$

We demonstrated that:

- The division into three time intervals that we have used covers all possible cases for τ: eq.(1);
- In each of these temporal intervals, there is always a version of the *Always* operator which is `True` whatever τ: eqs. (42) to (44).

We have therefore shown that the completeness property 2 is satisfied by the *Always* semantics for any value of τ. □

4.3 Disjointness

Proof. Let us come back to the exclusivity property 3:

$$\begin{aligned} & \neg[(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \vee (\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \vee (\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau)] \\ \Longleftrightarrow & \neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \wedge \neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \wedge \neg(\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau) \end{aligned} \tag{45}$$

Let us consider separately the three different cases and show that they are satisfied by the *Always* semantics.

- $\neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau)$

$$\Longleftrightarrow \neg(\tau \geq t+b \wedge A \wedge \tau \geq t+a \wedge C)$$

using eqs. (39) and (41).

$$\Longleftrightarrow \neg(\tau \geq t+b \wedge A \wedge C)$$

by time simplification according to eq 1.

$$\Longleftrightarrow \neg\left(\tau \geq t+b \wedge \left(\bigwedge \begin{array}{l} (\forall t_1 \in [t+a, t+b], (\mathcal{X}, t_1) \vDash \varphi) \\ (\exists t_3 \in [t+a, t+b] : (\mathcal{X}, t_3) \vDash \neg\varphi) \end{array}\right)\right)$$

simplifying the time intervals according to eq 1 and using eqs. (36) and (38).

$$\Longleftrightarrow \neg\left(\tau \geq t+b \wedge \left(\bigwedge \begin{array}{c} (\forall t_1 \in [t+a, t+b], (\mathcal{X}, t_1) \vDash \varphi) \\ \neg(\forall t_3 \in [t+a, t+b], (\mathcal{X}, t_3) \vDash \varphi) \end{array}\right)\right)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\begin{aligned} & \Longleftrightarrow \neg \mathit{False} \\ & \Longleftrightarrow \mathit{True} \end{aligned} \tag{46}$$

- $\neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau)$

$$\Longleftrightarrow \neg\left(\bigwedge \begin{array}{l} (\tau \geq t+b \wedge A) \\ (\tau < t+a \vee (\tau < t+b \wedge B)) \end{array}\right)$$

using eqs. (39) and (40).

$$\Longleftrightarrow \neg \mathit{False}$$

by conjunction on time intervals

$$\Longleftrightarrow \mathit{True} \tag{47}$$

- $\neg(\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau)$

$$\iff \neg\left(\bigvee \begin{array}{l} (\tau < t+a \wedge \tau \geq t+a \wedge C) \\ (\tau < t+b \wedge B \wedge \tau \geq t+a \wedge C) \end{array}\right)$$

using eqs. (40) and (41).

$$\iff \neg\left(\bigwedge \begin{array}{l} (\tau \geq t+a \wedge \tau < t+b) \\ (\forall t_2 \in [t+a,\tau], (\mathcal{X},t_2) \vDash \varphi) \\ (\exists t_3 \in [t+a,\tau] : (\mathcal{X},t_3) \vDash \neg\varphi) \end{array}\right)$$

simplifying by conjunction on time intervals and using eqs. (37) and (38).

$$\iff \neg\left(\bigwedge \begin{array}{c} (\tau \geq t+a \wedge \tau < t+b) \\ (\forall t_2 \in [t+a,\tau], (\mathcal{X},t_2) \vDash \varphi) \\ \neg(\forall t_3 \in [t+a,\tau], (\mathcal{X},t_3) \vDash \varphi) \end{array}\right)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\iff \neg False$$

$$\iff True \tag{48}$$

Now, we can comeback to the beginning equation 4.3:

$$\neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \wedge \neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \wedge \neg(\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau) \tag{49}$$

based on the results of eqs. (46) to (48), we obtain:

$$\begin{aligned} &\iff True \wedge True \wedge True \\ &\iff True \end{aligned} \tag{50}$$

We showed that the disjointness condition is satisfied by the *Always* semantics.

□

4.4 Equivalence between offline and online logic

Proof.

$$\begin{aligned} \tau \geq t+b &\implies ((\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau) \wedge (\neg\mathbf{B}_P \Leftrightarrow \mathbf{F}_P^\tau)) \\ \iff (\tau \geq t+b &\implies (\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau)) \wedge (\tau \geq t+b \implies (\neg\mathbf{B}_P \Leftrightarrow \mathbf{F}_P^\tau)) \end{aligned} \tag{51}$$

Let us consider that $\tau \geq t+b$. Let us show that it implies that $(\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau)$ and $(\neg\mathbf{B}_P \Leftrightarrow \mathbf{F}_P^\tau)$ are *True*.

- $\tau \geq t+b \implies (\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau)$
  $\mathbf{B}_P$ corresponds exactly to A (eq 36). According to eq 39, equivalence is obtained directly.

- $\tau \geq t+b \implies (\neg \mathbf{B}_P \Leftrightarrow \mathbf{F}_P^\tau)$

$$\Longleftrightarrow \neg(\forall t' \in [t+a,t+b] : (\mathcal{X},t') \models \varphi) \Leftrightarrow (\exists t_3 \in [t+a,t+b] : (\mathcal{X},t_3) \models \neg\varphi)$$

simplifying time intervals according to eq 1 and using eqs. (31) and (32)

$$\Longleftrightarrow \neg(\forall t' \in [t+a,t+b] : (\mathcal{X},t') \models \varphi) \Leftrightarrow \neg(\forall t_3 \in [t+a,t+b] : (\mathcal{X},t_3) \models \varphi)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\Longleftrightarrow True \tag{52}$$

We demonstrate in section 4.4 that both parts of eq 51 are satisfied. Thus, we show that equivalence between offline and online versions is respected by Always semantics, from time $t+b$. □

4.5 Immutability: Positive and negative logics are final

Proof.

- $\exists t \in \mathbb{T} : \mathbf{T}_\varphi^t \implies \forall t' \geq t, \mathbf{T}_\varphi^{t'}$

  Let us come back to the positive logic equation:

$$\tau \geq t+b \wedge \forall t_1 \in [t+a,t+b], (\mathcal{X},t_1) \models \varphi \tag{53}$$

  $\tau \geq t+b$ part is directly proven according to Remark 3.

  $\forall t_1 \in [t+a,t+b], (\mathcal{X},t_1) \models \varphi$ is not affected by the current time so if the property was already True at a time t', it is still the case at time $t'+1$.

- $\exists t \in \mathbb{T} : \mathbf{F}_\varphi^t \implies \forall t' \geq t, \mathbf{F}_\varphi^{t'}$

  Directly proven according to Remark 3.

We showed that whatever the time t', if T or F is satisfied, it is still the case at time $t'+1$ □

5 Until

5.1 Until definitions

Until offline:

$$(\mathcal{X},t) \models \varphi_1 \mathcal{U}_{[a,b]} \varphi_2 \Leftrightarrow \exists t' \in [t+a,t+b] : (\mathcal{X},t') \models \varphi_2 \wedge \forall t'' \in [t,t'], (\mathcal{X},t'') \models \varphi_1 \tag{54}$$

Until online: $\varphi_1 \mathcal{U}_{[a,b]} \varphi_2$

$$\begin{aligned}
\mathbf{T}_P^\tau \quad & \tau \geq t+a \wedge \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi_2 \quad \wedge \\
& \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1 && (55) \\
\mathbf{U}_P^\tau \quad & ((\tau < t+a) \wedge \forall t_3 \in [t, \tau], (\mathcal{X}, t_3) \vDash \varphi_1) \vee (\tau \geq t+a \wedge \tau < t+b \wedge \\
& \forall t_4 \in [t, \tau], (\mathcal{X}, t_4) \vDash \varphi_1 \wedge \forall t_5 \in [t+a, \tau], (\mathcal{X}, t_5) \vDash \neg\varphi_2) && (56) \\
\mathbf{F}_P^\tau \quad & (\exists t_6 \in [t, min(\tau, t+a)] : (\mathcal{X}, t_6) \vDash \neg\varphi_1) \quad \vee \\
& (\tau \geq t+a \wedge \tau < t+b \wedge \exists t_7 \in [t+a, \tau] : (\mathcal{X}, t_7) \vDash \neg\varphi_1 \wedge \\
& \neg(\exists t_8 \in [t+a, \tau] : (\mathcal{X}, t_8) \vDash \varphi_2 \wedge \forall t_9 \in [t, t_8], (\mathcal{X}, t_9) \vDash \varphi_1)) \quad \vee \\
& (\tau \geq t+b \wedge \\
& \neg(\exists t_{10} \in [t+a, t+b] : (\mathcal{X}, t_{10}) \vDash \varphi_2 \wedge \forall t_{11} \in [t, t_{10}], (\mathcal{X}, t_{11}) \vDash \varphi_1)) && (57)
\end{aligned}$$

Let A, B, C, D and E such that:

$$\begin{aligned}
& A = \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi_2 \wedge \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1 && (58) \\
& B = \forall t_3 \in [t, \tau], (\mathcal{X}, t_3) \vDash \varphi_1) && (59) \\
& C = \forall t_4 \in [t, \tau], (\mathcal{X}, t_4) \vDash \varphi_1 \wedge \forall t_5 \in [t+a, \tau], (\mathcal{X}, t_5) \vDash \neg\varphi_2) && (60) \\
& D = \exists t_6 \in [t, min(\tau, t+a)] : (\mathcal{X}, t_6) \vDash \neg\varphi_1 && (61) \\
& E = \exists t_7 \in [t+a, \tau] : (\mathcal{X}, t_7) \vDash \neg\varphi_1 && (62)
\end{aligned}$$

We obtain:

$$\begin{aligned}
\mathbf{T}_P^\tau \quad & \tau \geq t+a \wedge A && (63) \\
\mathbf{U}_P^\tau \quad & (\tau < t+a \wedge B) \vee (\tau \geq t+a \wedge \tau < t+b \wedge C) && (64) \\
\mathbf{F}_P^\tau \quad & D \vee (\tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A) \vee (\tau \geq t+b \wedge \neg A) && (65)
\end{aligned}$$

5.2 Completeness

Proof. Let us separate the consider time interval in three different sections as introduced in the property 1. Let us show that the property is satisfied for each of these three intervals.

1. **Starting hypothesis:** $\tau < t+a$

$$\tau < t+a \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau)$$

$$\iff \tau < t+a \wedge \left(\bigvee \begin{array}{l} \tau \geq t+a \wedge A \\ \tau < t+a \wedge B \\ \tau \geq t+a \wedge \tau < t+b \wedge C \\ D \\ \tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A \\ \tau \geq t+b \wedge \neg A \end{array} \right)$$

using eqs. (63) to (65).

$$\iff \tau < t + a \wedge (B \vee D)$$

simplifying and removing false cases in disjunction

$$\iff \left(\bigvee \begin{array}{l} \forall t_3 \in [t, \tau], (\mathcal{X}, t_3) \models \varphi_1 \\ \exists t_6 \in [t, \tau] : (\mathcal{X}, t_6) \models \neg\varphi_1 \end{array} \right)$$

simplifying the time intervals and using eqs. (59) and (61).

$$\iff \left(\bigvee \begin{array}{c} \forall t_3 \in [t, \tau], (\mathcal{X}, t_3) \models \varphi_1 \\ \neg\forall t_6 \in [t, \tau] : (\mathcal{X}, t_6) \models \varphi_1 \end{array} \right)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\iff True \tag{66}$$

2. **Starting hypothesis:** $\tau \geq t + a \wedge \tau < t + b$

$$\tau \geq t + a \wedge \tau < t + b \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau)$$

$$\iff \tau \geq t + a \wedge \tau < t + b \wedge \left(\bigvee \begin{array}{l} \tau \geq t + a \wedge A \\ \tau < t + a \wedge B \\ \tau \geq t + a \wedge \tau < t + b \wedge C \\ D \\ \tau \geq t + a \wedge \tau < t + b \wedge E \wedge \neg A \\ \tau \geq t + b \wedge \neg A \end{array} \right)$$

using eqs. (63) to (65).

$$\iff \tau \geq t + a \wedge \tau < t + b \wedge (A \vee C \vee D \vee (E \wedge \neg A))$$

simplifying and removing false cases in disjunction

$$\iff \left(\bigvee \begin{array}{l} \exists t_1 \in [t + a, \tau] : (\mathcal{X}, t_1) \models \varphi_2 \wedge \\ \qquad \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \models \varphi_1 \\ \forall t_4 \in [t, \tau], (\mathcal{X}, t_4) \models \varphi_1 \wedge \forall t_5 \in [t + a, \tau], (\mathcal{X}, t_5) \models \neg\varphi_2 \\ \exists t_6 \in [t, t + a] : (\mathcal{X}, t_6) \models \neg\varphi_1 \\ \exists t_7 \in [t + a, \tau] : (\mathcal{X}, t_7) \models \neg\varphi_1 \wedge \\ \qquad \neg(\exists t_1 \in [t + a, \tau] : (\mathcal{X}, t_1) \models \varphi_2 \wedge \\ \qquad\quad \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \models \varphi_1) \end{array} \right)$$

simplifying the time intervals and using eqs. (58) and (60) to (62).

Let's consider the following hypothesis, and then its opposite:

- $\forall t_{11} \in [t, \tau], (\mathcal{X}, t_{11}) \models \varphi_1 \iff \neg\exists t_{11} \in [t, \tau] : (\mathcal{X}, t_{11}) \models \neg\varphi_1$:

$$\iff \left(\bigvee \begin{array}{l} \exists t_1 \in [t + a, \tau] : (\mathcal{X}, t_1) \models \varphi_2 \\ \forall t_5 \in [t + a, \tau], (\mathcal{X}, t_5) \models \neg\varphi_2 \end{array} \right)$$

simplifying and removing false cases in disjunction according to the hypothesis

$$\iff \left(\bigvee \begin{array}{l} \neg\forall t_1 \in [t+a,\tau], (\mathcal{X}, t_1) \vDash \neg\varphi_2 \\ \forall t_5 \in [t+a,\tau], (\mathcal{X}, t_5) \vDash \neg\varphi_2 \end{array} \right)$$

$$\iff True$$

- $\neg\forall t_{11} \in [t,\tau], (\mathcal{X}, t_{11}) \vDash \varphi_1 \iff \exists t_{11} \in [t,\tau] : (\mathcal{X}, t_{11}) \vDash \neg\varphi_1$:

$$\iff \left(\bigvee \begin{array}{l} A \\ \exists t_6 \in [t, t+a] : (\mathcal{X}, t_6) \vDash \neg\varphi_1 \\ \exists t_7 \in [t+a, \tau] : (\mathcal{X}, t_7) \vDash \neg\varphi_1 \wedge \neg A \end{array} \right)$$

Again, let's divide our proof in two sub-goals:

- $\exists t_6 \in [t, t+a] : (\mathcal{X}, t_6) \vDash \neg\varphi_1$:

$$\iff True$$

simplifying and removing false cases in disjunction according to the hypothesis

- $\neg\exists t_6 \in [t, t+a] : (\mathcal{X}, t_6) \vDash \neg\varphi_1$:

$$\iff A \vee \neg A$$

simplifying the time intervals and removing false cases in disjunction according to the hypothesis

$$\iff True \tag{67}$$

All sub-goals have been proved, so the completeness of Until for τ such that $\tau \geq t+a \wedge \tau < t+b$.

3. **Starting hypothesis:** $\tau \geq t+b$

$$\tau \geq t+b \wedge (\mathbf{T}_P^\tau \vee \mathbf{U}_P^\tau \vee \mathbf{F}_P^\tau)$$

$$\iff \tau \geq t+b \wedge \left(\bigvee \begin{array}{l} \tau \geq t+a \wedge A \\ \tau < t+a \wedge B \\ \tau \geq t+a \wedge \tau < t+b \wedge C \\ D \\ \tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A \\ \tau \geq t+b \wedge \neg A \end{array} \right)$$

using eqs. (63) to (65).

$$\iff \tau \geq t+b \wedge (A \vee D \vee \neg A)$$

simplifying and removing false cases in disjunction

$$\iff True \tag{68}$$

We demonstrated that:

- The division into three time intervals that we have used covers all possible cases for τ: eq.(1);
- In each of these temporal intervals, there is always a version of the *Until* operator which is `True` whatever τ: eqs. (66) to (68).

We have therefore shown that the completeness property 2 is satisfied by the *Until* semantics for any value of τ.

□

5.3 Disjointness

Proof. Let us come back to the exclusivity property 3:

$$\begin{aligned} &\neg[(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \vee (\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \vee (\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau)] \\ \Longleftrightarrow\ &\neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \wedge \neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \wedge \neg(\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau) \end{aligned} \tag{69}$$

Let us consider separately the three different cases and show that they are satisfied by the *Until* semantics.

- $\neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau)$

$$\Longleftrightarrow \neg\big((\tau \geq t+a \wedge A) \wedge ((\tau < t+a \wedge B) \vee (\tau \geq t+a \wedge \tau < t+b \wedge C))\big)$$

using eqs. (63) and (64).

$$\Longleftrightarrow \neg\left(\bigvee \begin{array}{l} \tau \geq t+a \wedge A \wedge \tau < t+a \wedge B \\ \tau \geq t+a \wedge A \wedge \tau \geq t+a \wedge \tau < t+b \wedge C \end{array}\right)$$

$$\Longleftrightarrow \neg\left(\bigwedge \begin{array}{l} \tau \geq t+a \wedge \tau < t+b \\ \exists t_1 \in [t+a,\tau] : (\mathcal{X},t_1) \vDash \varphi_2 \\ \forall t_2 \in [t,t_1], (\mathcal{X},t_2) \vDash \varphi_1 \\ \forall t_4 \in [t,\tau], (\mathcal{X},t_4) \vDash \varphi_1 \\ \forall t_5 \in [t+a,\tau], (\mathcal{X},t_5) \vDash \neg\varphi_2 \end{array}\right)$$

simplifying the time intervals and removing false cases in disjunction.

$$\Longleftrightarrow \neg\left(\bigwedge \begin{array}{l} \tau \geq t+a \wedge \tau < t+b \\ \neg\forall t_1 \in [t+a,\tau] : (\mathcal{X},t_1) \vDash \neg\varphi_2 \\ \quad\forall t_5 \in [t+a,\tau], (\mathcal{X},t_5) \vDash \neg\varphi_2 \\ \quad\forall t_2 \in [t,t_1], (\mathcal{X},t_2) \vDash \varphi_1 \\ \quad\forall t_4 \in [t,\tau], (\mathcal{X},t_4) \vDash \varphi_1 \end{array}\right)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\Longleftrightarrow \neg False$$

by conjunction.

$$\Longleftrightarrow True \tag{70}$$

- $\neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau)$

$$\begin{aligned} \Longleftrightarrow \neg\big(&(\tau \geq t+a \wedge A) \wedge \\ &(D \vee (\tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A) \vee (\tau \geq t+b \wedge \neg A))\big) \end{aligned}$$

using eqs. (63) and (65).

$$\Longleftrightarrow \neg\left(\bigvee \begin{array}{l} \tau \geq t+a \wedge A \wedge D \\ \tau \geq t+a \wedge A \wedge \tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A \\ \tau \geq t+a \wedge A \wedge \tau \geq t+b \wedge \neg A \end{array}\right)$$

$$\Longleftrightarrow \neg \quad \tau \geq t+a \wedge A \wedge D$$

removing false cases in disjunction

$$\iff \neg \left(\bigwedge \begin{array}{l} \tau \geq t+a \\ \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \models \varphi_2 \\ \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \models \varphi_1 \\ \exists t_6 \in [t, min(\tau, t+a)] : (\mathcal{X}, t_6) \models \neg \varphi_1 \end{array} \right)$$

simplifying and using eqs. (58) and (61).

$$\iff \neg \left(\bigwedge \begin{array}{l} \tau \geq t+a \\ \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \models \varphi_2 \\ \forall t_2 \in [t, t+a], (\mathcal{X}, t_2) \models \varphi_1 \\ \forall t_{2b} \in [t+a, t_1], (\mathcal{X}, t_{2b}) \models \varphi_1 \\ \neg \forall t_6 \in [t, t+a] : (\mathcal{X}, t_6) \models \varphi_1 \end{array} \right)$$

developing time intervals and replacing $\exists Q$ by $\neg \forall \neg Q$.

$$\iff \neg False$$

by conjunction.

$$\iff True \tag{71}$$

- $\neg(\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau)$

$$\iff \neg\big(((\tau < t+a \wedge B) \vee (\tau \geq t+a \wedge \tau < t+b \wedge C)) \wedge \\ (D \vee (\tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A) \vee (\tau \geq t+b \wedge \neg A))\big)$$

using eqs. (64) and (65).

$$\iff \neg \left(\bigvee \begin{array}{l} \tau < t+a \wedge B \wedge D \\ \tau < t+a \wedge B \wedge \tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A \\ \tau < t+a \wedge B \wedge \tau \geq t+b \wedge \neg A \\ \tau \geq t+a \wedge \tau < t+b \wedge C \wedge D \\ \tau \geq t+a \wedge \tau < t+b \wedge C \wedge \tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A \\ \tau \geq t+a \wedge \tau < t+b \wedge C \wedge \tau \geq t+b \wedge \neg A \end{array} \right)$$

$$\iff \neg \left(\bigvee \begin{array}{l} \tau < t+a \wedge B \wedge D \\ \tau \geq t+a \wedge \tau < t+b \wedge C \wedge D \\ \tau \geq t+a \wedge \tau < t+b \wedge C \wedge E \wedge \neg A \end{array} \right)$$

simplifying and removing false cases in disjunction.

Satisfaction of the property means that each of these 3 cases should be false. Let's consider them separately:

- $\tau < t+a \wedge B \wedge D$:

$$\iff \left(\bigwedge \begin{array}{l} \tau < t+a \\ \forall t_3 \in [t, \tau], (\mathcal{X}, t_3) \models \varphi_1 \\ \exists t_6 \in [t, \tau] : (\mathcal{X}, t_6) \models \neg \varphi_1 \end{array} \right)$$

simplifying and using eqs. (59) and (61).

$$\iff \left(\bigwedge \begin{array}{l} \tau < t+a \\ \forall t_3 \in [t,\tau], (\mathcal{X}, t_3) \models \varphi_1 \\ \neg \forall t_6 \in [t,\tau] : (\mathcal{X}, t_6) \models \varphi_1 \end{array} \right)$$

replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\iff False$$

- $\tau \geq t+a \wedge \tau < t+b \wedge C \wedge D$:

$$\iff \left(\bigwedge \begin{array}{l} \tau \geq t+a \wedge \tau < t+b \\ \forall t_4 \in [t,\tau], (\mathcal{X}, t_4) \models \varphi_1 \\ \forall t_5 \in [t+a,\tau], (\mathcal{X}, t_5) \models \neg\varphi_2 \\ \exists t_6 \in [t, min(\tau, t+a)] : (\mathcal{X}, t_6) \models \neg\varphi_1 \end{array} \right)$$

simplifying and using eqs. (60) and (61).

$$\iff \left(\bigwedge \begin{array}{l} \tau \geq t+a \wedge \tau < t+b \\ \neg \forall t_6 \in [t,t+a] : (\mathcal{X}, t_6) \models \varphi_1 \\ \forall t_4 \in [t,t+a], (\mathcal{X}, t_4) \models \varphi_1 \\ \forall t_{4b} \in [t+a,\tau], (\mathcal{X}, t_{4b}) \models \varphi_1 \\ \forall t_5 \in [t+a,\tau], (\mathcal{X}, t_5) \models \neg\varphi_2 \end{array} \right)$$

developing time intervals and replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\iff False$$

- $\tau \geq t+a \wedge \tau < t+b \wedge C \wedge E \wedge \neg A$:

$$\iff \left(\bigwedge \begin{array}{l} \forall t_4 \in [t,\tau], (\mathcal{X}, t_4) \models \varphi_1 \\ \forall t_5 \in [t+a,\tau], (\mathcal{X}, t_5) \models \neg\varphi_2 \\ \exists t_7 \in [t+a,\tau] : (\mathcal{X}, t_7) \models \neg\varphi_1 \\ \neg(\exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \models \varphi_2 \wedge \\ \quad \forall t_2 \in [t,t_1], (\mathcal{X}, t_2) \models \varphi_1) \end{array} \right)$$

simplifying and using eqs. (58), (60) and (62).

$$\iff \left(\bigwedge \begin{array}{l} \forall t_4 \in [t,t+a], (\mathcal{X}, t_4) \models \varphi_1 \\ \forall t_{4b} \in [t+a,\tau], (\mathcal{X}, t_{4b}) \models \varphi_1 \\ \neg \forall t_7 \in [t+a,\tau] : (\mathcal{X}, t_7) \models \varphi_1 \\ \forall t_5 \in [t+a,\tau], (\mathcal{X}, t_5) \models \neg\varphi_2 \\ \neg(\exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \models \varphi_2 \wedge \\ \quad \forall t_2 \in [t,t_1], (\mathcal{X}, t_2) \models \varphi_1) \end{array} \right)$$

developing time intervals and replacing $\exists Q$ by $\neg\forall\neg Q$.

$$\iff False \tag{72}$$

Each case of the previous disjunction are false. We obtain $\neg False \iff True$; the property is satisfied for this temporal interval.

Now, we can comeback to the beginning equation 4.3:

$$\neg(\mathbf{T}_P^\tau \wedge \mathbf{F}_P^\tau) \wedge \neg(\mathbf{T}_P^\tau \wedge \mathbf{U}_P^\tau) \wedge \neg(\mathbf{U}_P^\tau \wedge \mathbf{F}_P^\tau) \tag{73}$$

based on the results of eqs. (70) to (72), we obtain:

$$\begin{aligned} &\iff True \wedge True \wedge True \\ &\iff True \end{aligned} \tag{74}$$

We showed that the disjointness condition is satisfied by the *Until* semantics.

□

5.4 Equivalence between offline and online logic

Proof.

$$\begin{aligned} &\tau \geq t+b \implies ((\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau) \wedge (\neg \mathbf{B}_P \Leftrightarrow \mathbf{F}_P^\tau)) \\ \iff &(\tau \geq t+b \implies (\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau)) \wedge (\tau \geq t+b \implies (\neg \mathbf{B}_P \Leftrightarrow \mathbf{F}_P^\tau)) \end{aligned} \tag{75}$$

Let us consider that $\tau \geq t+b$. Let us show that it implies that $(\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau)$ and $(\neg \mathbf{B}_P \Leftrightarrow \mathbf{F}_P^\tau)$ are $True$.

- $\tau \geq t+b \implies (\mathbf{B}_P \Leftrightarrow \mathbf{T}_P^\tau)$

$$\begin{aligned} \iff &\tau \geq t+b \wedge \big(\exists t' \in [t+a, t+b] : (\mathcal{X}, t') \models \varphi_2 \quad \wedge \\ &\forall t'' \in [t, t'], (\mathcal{X}, t'') \models \varphi_1\big) \Leftrightarrow \\ &\tau \geq t+b \wedge \big(\tau \geq t+a \wedge \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \models \varphi_2 \wedge \\ &\forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \models \varphi_1\big) \end{aligned}$$

using eqs. (54) and (63).

$$\begin{aligned} \iff &\tau \geq t+b \wedge \big(\exists t' \in [t+a, t+b] : (\mathcal{X}, t') \models \varphi_2 \quad \wedge \\ &\forall t'' \in [t, t'], (\mathcal{X}, t'') \models \varphi_1\big) \Leftrightarrow \\ &\tau \geq t+b \wedge \big(\exists t_1 \in [t+a, t+b] : (\mathcal{X}, t_1) \models \varphi_2 \quad \wedge \\ &\forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \models \varphi_1\big) \end{aligned}$$

simplifying time intervals

$$\iff True \tag{76}$$

- $\tau \geq t+b \implies (\neg \mathbf{B}_P \Leftrightarrow \mathbf{F}_P^{\tau})$

$$\iff \neg\big(\tau \geq t+b \wedge \big(\exists t' \in [t+a, t+b] : (\mathcal{X}, t') \vDash \varphi_2 \quad \wedge$$
$$\forall t'' \in [t, t'], (\mathcal{X}, t'') \vDash \varphi_1\big)\big) \Leftrightarrow$$
$$D \vee (\tau \geq t+b \wedge \neg A)$$

using eqs. (54) and (65).

$$\iff \neg A \Leftrightarrow D \vee \neg A$$

This proposition is true only if $(\neg(\neg A) \implies \neg D) \iff (A \implies \neg D)$

- $A \implies \neg D$

$$\iff \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi_2 \quad \wedge$$
$$\forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1 \implies$$
$$\neg\big(\exists t_6 \in [t, min(\tau, t+a)] : (\mathcal{X}, t_6) \vDash \neg \varphi_1\big)$$

using eqs. (54) and (65).

$$\iff \exists t_1 \in [t+a, t+b] : (\mathcal{X}, t_1) \vDash \varphi_2 \quad \wedge$$
$$\forall t_2 \in [t, t+a], (\mathcal{X}, t_2) \vDash \varphi_1 \wedge \forall t_2 \in [t+a, t_1], (\mathcal{X}, t_2) \vDash \varphi_1 \implies$$
$$\forall t_6 \in [t, t+a] : (\mathcal{X}, t_6) \vDash \varphi_1$$

replacing $\exists Q$ by $\neg \forall \neg Q$, simplifying and developing time intervals.

$$\iff True \tag{77}$$

We demonstrate in section 5.4 that both parts of eq 51 are satisfied. Thus, we show that equivalence between offline and online versions is respected by Until semantics, from time $t+b$. □

5.5 Immutability: Positive and negative logics are final

Proof.

- $\exists t \in \mathbb{T} : \mathbf{T}_{\varphi}^{t} \implies \forall t' \geq t, \mathbf{T}_{\varphi}^{t'}$

Let's remind the positive version of Until:

$$\tau \geq t+a \wedge \exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi_2 \wedge \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1$$

From remarks eqs. (8) and (9), we can directly conclude that equation 6 is satisfied.

- $\exists t \in \mathbb{T} : \mathbf{F}_{\varphi}^{t} \implies \forall t' \geq t, \mathbf{F}_{\varphi}^{t'}$

Let's remind the negative version of Until:

$$D \vee (\tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A) \vee (\tau \geq t+b \wedge \neg A)$$

We can divide this equation in three different cases.

- D
- $\tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A$
- $\tau \geq t+b \wedge \neg A$

Let's show that the equation 7 is satisfied for each of these three cases.

- $D \iff \exists t_6 \in [t, min(\tau, t+a)] : (\mathcal{X}, t_6) \vDash \neg\varphi_1$ This one is immediately satisfied according to 8.

- $\tau \geq t+a \wedge \tau < t+b \wedge E \wedge \neg A$

$$\begin{aligned} &\iff \tau \geq t+a \wedge \tau < t+b \wedge \exists t_7 \in [t+a, \tau] : (\mathcal{X}, t_7) \vDash \neg\varphi_1 \wedge \\ &\quad \neg(\exists t_1 \in [t+a, min(\tau, t+b)] : (\mathcal{X}, t_1) \vDash \varphi_2 \wedge \\ &\quad \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1) \\ &\iff \tau \geq t+a \wedge \tau < t+b \wedge True \wedge \neg(\exists t_1 \in [t+a, t_7] : (\mathcal{X}, t_1) \vDash \varphi_2 \wedge \\ &\quad \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1) \end{aligned}$$

by temporal simplification on time intervals and according to remark 8.

$\neg(\exists t_1 \in [t+a, t_7] : (\mathcal{X}, t_1) \vDash \varphi_2 \wedge \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1)$ do not depend of the τ value so if it is True at time t', it will still be True at time $t'+1$:

$$\iff \tau \geq t+a \wedge \tau < t+b \tag{78}$$

If $t'+1 \leq t+b$, this case is immediately satisfied. Else, we are in the case where $t'+1 = t+b$ and $t' < t+b$. This case is addressed by the next point.

- $t' \geq t+b \wedge \neg A$ or $t'+1 \geq t+b \wedge \neg A$

$$\iff \exists t_1 \in [t+a, t+b] : (\mathcal{X}, t_1) \vDash \varphi_2 \wedge \forall t_2 \in [t, t_1], (\mathcal{X}, t_2) \vDash \varphi_1 \tag{79}$$

by temporal simplification.

(80)

Result is not affected by the current time so if the property was already True at a time t', it is still the case at time $t'+1$.

We showed that whatever the time t', if $\mathbf{T}_P^{t'}$ or $\mathbf{F}_P^{t'}$ is satisfied, it is still True at time $t'+1$

□